Computing Rational Bisectors of Point/Surface and Sphere/Surface Pairs

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Given two objects, their bisector is defined as the set of points which are at equal distance from the two objects. Bisector construction plays an important role in many geometric computations: e.g., Voronoi diagrams' construction, medial axis transformation, shape decomposition, mesh generation, collision-avoidance motion planning, and NC tool path generation, to mention only a few.

Unfortunately, the bisector of even simple geometric primitives is not always simple. While the bisector of two lines in the plane is a line, the bisector of two skewed lines in $\mathbb{R}^3$ is a hyperbolic paraboloid of one sheet [1]. The bisector of two spheres of the same radius is a plane. However, the bisector of two spheres with different radii is a hyperboloid of two sheets or an ellipsoid (see the sidebar "Point-Sphere Bisector").

Freeform polynomial/rational primitives have bisectors which are significantly more complex than those for linear or circular primitives. For example, in the plane, the bisector of two cubic polynomial curves is an algebraic curve of degree 46 (see the sidebar "Degree of Planar Bisector Curve"). Moreover, the bisector curve is non-rational, in general. Because of these limitations, previous work has considered various approximation techniques for bisector curves in the plane. In particular, Farouki and Ramamurthy [5] developed an approximation algorithm that can reduce the approximation error within an arbitrary bound.

There are some special cases in which the bisector has a simple closed-form or a rational representation. Dutta and Hoffmann [1] considered the bisector of simple surfaces such as natural quadrics and toroidal surfaces. For these special types of surfaces, the bisector has a simple closed-form representation. Farouki and Johnstone [4] showed that the bisector of a point and a rational curve in the same plane is a rational curve. Elber and Kim [3] showed that the bisector of two rational space curves in $\mathbb{R}^3$ is a rational surface. Similarly, the bisector of a point and a rational space curve in $\mathbb{R}^3$ is a rational ruled surface [3].

This article shows that the bisector of a point and a rational surface in $\mathbb{R}^3$ is also a rational surface. This result implies that the bisector of a sphere and a surface with a rational offset is also a rational surface. Pottmann [9] classified the class of all rational curves and surfaces with rational offsets, which includes the Pythagorean Hodograph (PH) curves as an important subclass of all polynomial curves with rational offsets [6]. Simple surfaces (planes, spheres, cylinders, cones, tori), Dupin cyclides, rational canal surfaces (see discussion in the next paragraph), and non-developable rational ruled surfaces all belong to this class of rational surfaces with rational offsets [2, 8, 10]. Consequently, our result can be applied to a wide variety of rational curves/surfaces for the construction of their rational bisector curves/surfaces with circles/spheres.

A canal surface is defined as the sweep envelope surface of a moving sphere (possibly with a variable radius). Given a rational trajectory curve for the center of the sphere and a rational radius function of the sphere, Peternell and Pottmann [8] recently presented a surprising result that the canal surface is rational. The offset of a canal surface is another canal surface that is obtained by simply increasing the radius function by the offset distance and sweeping the enlarged sphere along the same trajectory. Consequently, rational canal surfaces (defined by rational trajectory curves and rational radius functions) are closed under offset operation and admit rational representations for their bisector surfaces with arbitrary spheres.

The rest of this article is organized as follows. We begin with the construction of a rational bisector surface of a point and a rational surface. The result is then extended to that for a rational bisector surface of a sphere and a rational surface with rational offsets. The implementation of the proposed algorithm as well

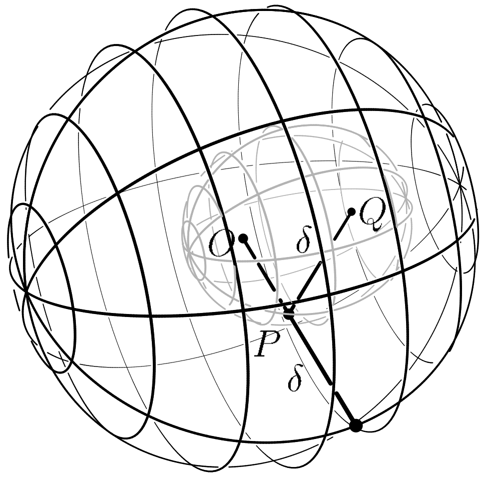
Point-Sphere Bisector

Let $S_r(O) \subset \mathbb{R}^3$ be a sphere of radius r with its center at O. The bisector of a point $Q \in \mathbb{R}^3$ and a sphere $S_r(O)$ is either an ellipsoid or a hyperboloid of two sheets, depending on whether Q is contained in the interior or in the exterior of the sphere $S_r(O)$. Note that each point P on the bisector surface satisfies

$$\delta = d(P, S_r(O)) = \|P - Q\|,$$

for some $\delta \geq 0$, where $d(P, S_r(O))$ is the distance between P and $S_r(O)$.

Case I: Q is inside the sphere $S_r(O)$ (see also Figure 2 (a)):

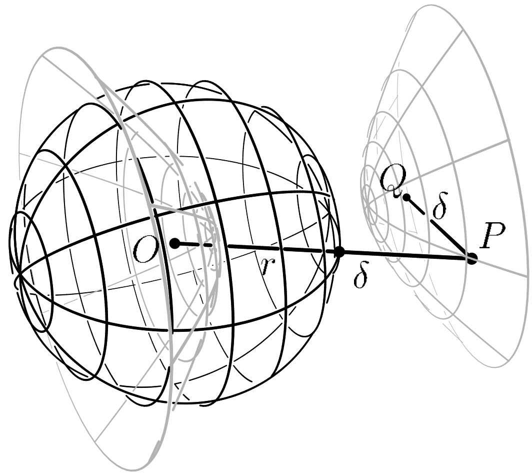


In the configuration shown above, we have

$$\begin{aligned} & \|P - O\| + \|P - Q\| \\ = \quad & (r - \delta) + \delta \\ = \quad & r, \end{aligned}$$

which means that, for each point P on the bisector, the sum of its distances to O and Q is constant. Hence, the point P is on an ellipsoid. Moreover, O and Q are the two focal points of the ellipsoid.

Case II: Q is outside the sphere $S_r(O)$ (see also Figure 2 (b)):



In the configuration shown above, we have

$$\begin{aligned} & \|P - O\| - \|P - Q\| \\ = \quad & (r + \delta) - \delta \\ = \quad & r, \end{aligned}$$

which means that, for each point P on the bisector, the difference of its distances to O and Q is constant. Hence, the point P is a hyperboloid of two sheets. Moreover, O and Q are the two focal points of the hyperboloid.
Note that the bisector of two spheres $S_{r_1}(O_1)$ and $S_{r_2}(O_2)$, $r_1 < r_2$, is identical to the bisector of the point O_1 and a sphere $S_{r_2 - r_1}(O_2)$.

Degree of Planar Bisector Curves

Given two planar polynomial curves $C_1(t) = (a(t), b(t))$ and $C_2(s) = (\alpha(s), \beta(s))$ of degrees m_1 and m_2, respectively, the point $P = (x, y)$ on the bisector curve of $C_1(t)$ and $C_2(s)$ satisfies the following three polynomial equations [3]:

$$\begin{aligned} \left\langle P - C_1(t), C_1'(t) \right\rangle &= 0, \\ \left\langle P - C_2(s), C_2'(s) \right\rangle &= 0, \\ \left\langle P - \frac{C_1(t) + C_2(s)}{2}, C_1(t) - C_2(s) \right\rangle &= 0. \end{aligned}$$

That is, we have

$$\begin{aligned} a'(t)x + b'(t)y &= a(t)a'(t) + b(t)b'(t), && (1) \\ \alpha'(s)x + \beta'(s)y &= \alpha(s)\alpha'(s) + \beta(s)\beta'(s), && (2) \\ e(s,t)x + f(s,t)y &= g(s,t), && (3) \end{aligned}$$

where $e(s,t) = a'(t) - \alpha'(s)$, $f(s,t) = b'(t) - \beta'(s)$, and $g(s,t) = \frac{1}{2}(a(t)^2 + b(t)^2 - \alpha(s)^2 - \beta(s)^2)$.
When we eliminate the variable t from Equations (1) and (3), we get an algebraic equation in three variables x, y, s:

$$F(x, y, s) = 0. \qquad (4)$$

By further eliminating the variable s from Equations (2) and (4), we get an algebraic equation in two variables x and y:

$$G(x, y) = 0, \qquad (5)$$

which represents the bisector curve of $C_1(t)$ and $C_2(s)$.
In the case of low degree curves $C_1(t)$ and $C_2(s)$, ($1 \leq m_1 \leq 3$ and $1 \leq m_2 \leq 6$), we have observed that the bisector curve of Equation (5) is an algebraic curve of degree $7m_1m_2 - 3(m_1 + m_2) + 1$, which is also irreducible over rational coefficients. For two cubic curves, the degree is thus 46, which is too high to be useful in practice.

as all the examples presented in this paper were created using IRIT [7], a solid modeling system developed at the Technion, Israel.

Bisector of a Point and a Surface

Let $Q \in \mathbb{R}^3$ be a fixed point and $S(u,v) \subset \mathbb{R}^3$ be a regular C^1-continuous parametric surface. Assume that $P(u,v)$ is a bisector point of Q and $S(u,v)$. Then, we have

$$(P(u,v) - S(u,v)) \;\|\; N(u,v), \qquad (6)$$

$$\|P(u,v) - Q\| = \|P(u,v) - S(u,v)\|, \qquad (7)$$

where $\|$ denotes the parallel relationship, $N(u,v)$ is the normal vector of $S(u,v)$, and $\|\cdot\|$ denotes the length of a vector.

Equation (6) can be rewritten using the following two constraints:

$$\begin{aligned} \left\langle P(u,v) - S(u,v), \frac{\partial S(u,v)}{\partial u} \right\rangle &= 0, \\ \left\langle P(u,v) - S(u,v), \frac{\partial S(u,v)}{\partial v} \right\rangle &= 0, \end{aligned} \qquad (8)$$

whereas Equation (7) can be rewritten as follows:

$$\begin{aligned} &\langle P(u,v) - Q, P(u,v) - Q \rangle \\ &= \langle P(u,v) - S(u,v), P(u,v) - S(u,v) \rangle, \end{aligned}$$

or equivalently,

$$\begin{aligned} &\langle P(u,v), S(u,v) - Q \rangle \\ &= \frac{\langle S(u,v), S(u,v) \rangle - \langle Q, Q \rangle}{2}. \end{aligned} \qquad (9)$$

The constraints in Equations (8)–(9) are all *linear* in $P(u,v)$. Using Cramer's rule, one can solve these equations for $P(u,v)$ and compute a rational surface representation of $P(u,v)$ provided that $S(u,v)$ is a rational surface (see the sidebar "Rational Parameterization").

Figure 1 (a) shows the bisector surface of a point and a plane, which is a parabolic surface of revolution. Figure 1 (b) considers the bisector surface of a point and a bicubic surface. The bisector surface has degree $(11, 11)$.

Figures 2–4 present several examples of the bisector surface between a point and a simple surface. The sphere is represented using a rational biquadratic surface. The cone and cylinder are represented using a rational NURBS surface of degree (1,2). The torus is represented as a biquadratic NURBS surface as well. The bisector of a point and a sphere turns out to be a rational parametric surface of degree $(16, 16)$. The bisector of a point and a cone (or a cylinder) is of degree $(8, 16)$. The bisector of a point and a torus is of degree $(16, 16)$.

Bisector of a Sphere and a Surface with Rational Offsets

Let $S_d(Q)$ denote the sphere of radius d with its center at Q, and a normal field pointing outside, and let $S_d(u,v)$ denote the offset surface of $S(u,v)$ by an offset radius d:

$$S_d(u,v) = S(u,v) + d\frac{N(u,v)}{\|N(u,v)\|},$$

where $N(u,v)$ is the normal vector field of $S(u,v)$. Note that $N(u,v)$ is the normal field of $S_d(u,v)$ as well, because the normal field is preserved under the offset operation.

Without loss of generality, we consider only the bisector points P along the positive normal directions: outside the sphere $S_d(Q)$ and along the oriented side of the normal direction $N(u,v)$ of $S_d(u,v)$. Then, the bisector surface of the sphere $S_d(Q)$ and the offset surface $S_d(u,v)$ is the same as the bisector surface of the point Q and the given surface $S(u,v)$. A point P at an equal distance r between $S_d(Q)$ and $S_d(u,v)$ is at an equal distance $r + d$ between Q and $S(u,v)$, and vice versa, for all d. Similarly, the bisector surface of the sphere $S_d(Q)$ and the surface $S(u,v)$ is the same as the bisector surface of the point Q and the offset surface $S_{-d}(u,v)$.

Assume that $S(u,v)$ is a rational surface, which has a rational offset surface $S_r(u,v)$, for some $r \neq 0$. Then the offset surface $S_{-d}(u,v)$ is rational, for all d, since we have

$$\begin{aligned} S_{-d}(u,v) &= S(u,v) - d\,\frac{N(u,v)}{\|N(u,v)\|}, \\ &= S(u,v) - \frac{d}{r}\,r\,\frac{N(u,v)}{\|N(u,v)\|}, \\ &= S(u,v) - \frac{d}{r}\left(S_r(u,v) - S(u,v)\right), \end{aligned}$$

where $S(u,v)$ and $S_r(u,v) - S(u,v)$ are clearly rational. Consequently, the bisector between $S_d(Q)$ and $S(u,v)$ is a rational surface if the surface $S(u,v)$ has a rational offset surface.

Simple surfaces (such as planes, spheres, cylinders, cones, and tori) are all rational. Moreover, they have rational offsets since the offsets of these simple surfaces are again simple surfaces of the same type. Therefore,

Rational Parameterization

Let

$$
\begin{aligned}
\frac{\partial S(u,v)}{\partial u} &= (a_{11}(u,v), a_{12}(u,v), a_{13}(u,v)),\\
\frac{\partial S(u,v)}{\partial v} &= (a_{21}(u,v), a_{22}(u,v), a_{23}(u,v)),\\
S(u,v) - Q &= (a_{31}(u,v), a_{32}(u,v), a_{33}(u,v)),\\
\left\langle S(u,v), \frac{\partial S(u,v)}{\partial u} \right\rangle &= b_1(u,v),\\
\left\langle S(u,v), \frac{\partial S(u,v)}{\partial v} \right\rangle &= b_2(u,v),\\
\frac{\|S(u,v)\|^2 - \|Q\|^2}{2} &= b_3(u,v),\\
P(u,v) &= (p_x(u,v), p_y(u,v), p_z(u,v)),
\end{aligned}
$$

then Equations (8)–(9) can be reformulated as follows:

$$
\begin{bmatrix} a_{11}(u,v) & a_{12}(u,v) & a_{13}(u,v) \\ a_{21}(u,v) & a_{22}(u,v) & a_{23}(u,v) \\ a_{31}(u,v) & a_{32}(u,v) & a_{33}(u,v) \end{bmatrix}
\begin{bmatrix} p_x(u,v) \\ p_y(u,v) \\ p_z(u,v) \end{bmatrix}
=
\begin{bmatrix} b_1(u,v) \\ b_2(u,v) \\ b_3(u,v) \end{bmatrix}.
$$

Note that $a_{i,j}(u,v)$ and $b_i(u,v)$, $(i,j = 1,2,3)$, are all rational functions if $S(u,v)$ is a rational surface. By Cramer's rule, we can obtain rational parameterization of $p_x(u,v)$, $p_y(u,v)$, and $p_z(u,v)$ as follows:

$$
p_x(u,v) = \frac{\begin{vmatrix} b_1(u,v) & a_{12}(u,v) & a_{13}(u,v) \\ b_2(u,v) & a_{22}(u,v) & a_{23}(u,v) \\ b_3(u,v) & a_{32}(u,v) & a_{33}(u,v) \end{vmatrix}}{\begin{vmatrix} a_{11}(u,v) & a_{12}(u,v) & a_{13}(u,v) \\ a_{21}(u,v) & a_{22}(u,v) & a_{23}(u,v) \\ a_{31}(u,v) & a_{32}(u,v) & a_{33}(u,v) \end{vmatrix}},
$$

$$
p_y(u,v) = \frac{\begin{vmatrix} a_{11}(u,v) & b_1(u,v) & a_{13}(u,v) \\ a_{21}(u,v) & b_2(u,v) & a_{23}(u,v) \\ a_{31}(u,v) & b_3(u,v) & a_{33}(u,v) \end{vmatrix}}{\begin{vmatrix} a_{11}(u,v) & a_{12}(u,v) & a_{13}(u,v) \\ a_{21}(u,v) & a_{22}(u,v) & a_{23}(u,v) \\ a_{31}(u,v) & a_{32}(u,v) & a_{33}(u,v) \end{vmatrix}},
$$

$$
p_z(u,v) = \frac{\begin{vmatrix} a_{11}(u,v) & a_{12}(u,v) & b_1(u,v) \\ a_{21}(u,v) & a_{22}(u,v) & b_2(u,v) \\ a_{31}(u,v) & a_{32}(u,v) & b_3(u,v) \end{vmatrix}}{\begin{vmatrix} a_{11}(u,v) & a_{12}(u,v) & a_{13}(u,v) \\ a_{21}(u,v) & a_{22}(u,v) & a_{23}(u,v) \\ a_{31}(u,v) & a_{32}(u,v) & a_{33}(u,v) \end{vmatrix}}.
$$

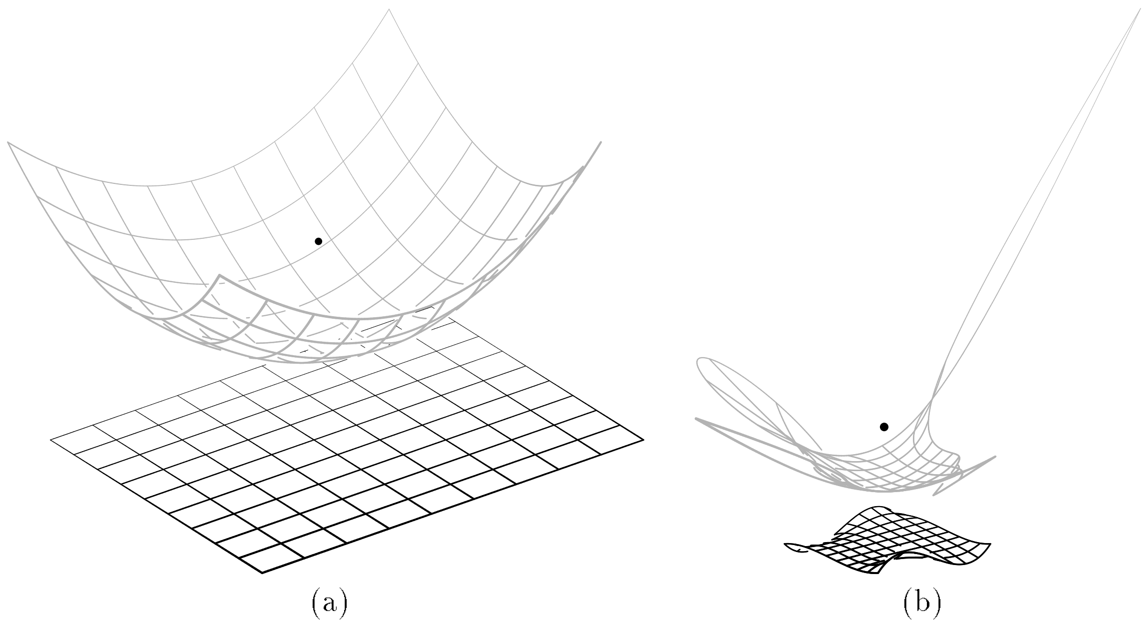


Figure 1: A bisector surface (in gray) of a point and a plane is shown in (a). A biquadratic surface is constructed that has the shape of a parabolic surface of revolution. In (b), the bisector surface (in gray) of a point and a general bicubic surface is shown.

one is able to compute the bisector surface between a sphere and a simple surface as a rational surface. Dupin cyclides and canal surfaces (defined by rational spine curves and rational radius functions) are also rational and closed under offset operation [2, 8]. Therefore, the bisector between a sphere and a surface (of these special types) also becomes a rational surface. Figure 5 shows an example of the bisector surface between a sphere and a canal surface.

Pottmann [9] classified the class of all rational curves

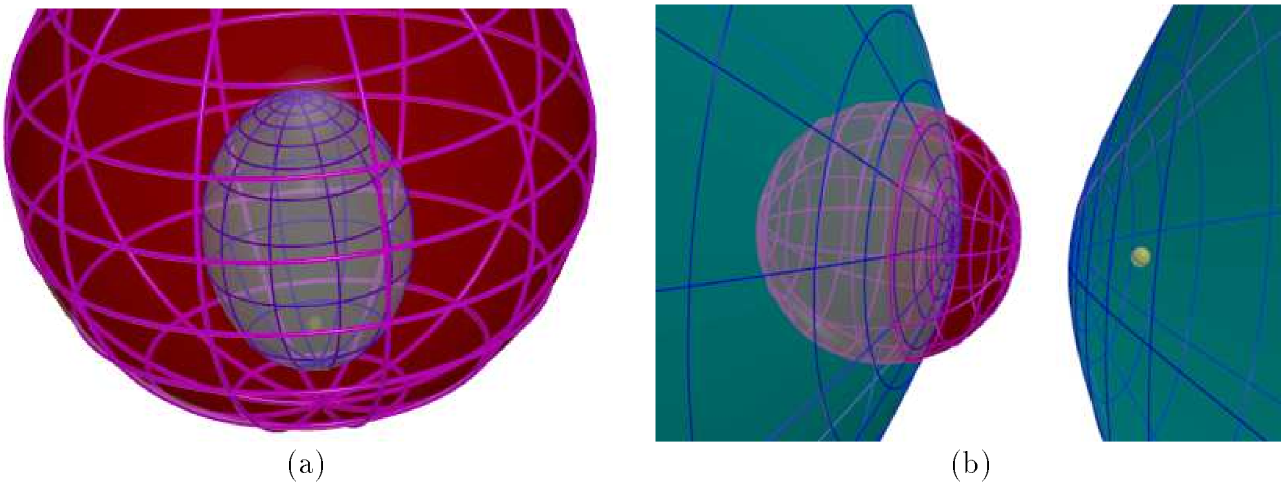


Figure 2: Two views of the bisector surface between a point and a sphere. In (a), the point is inside the sphere and the bisector surface is an ellipsoid. In (b), the point is outside the sphere and the bisector surface is a hyperboloid of two infinite sheets.

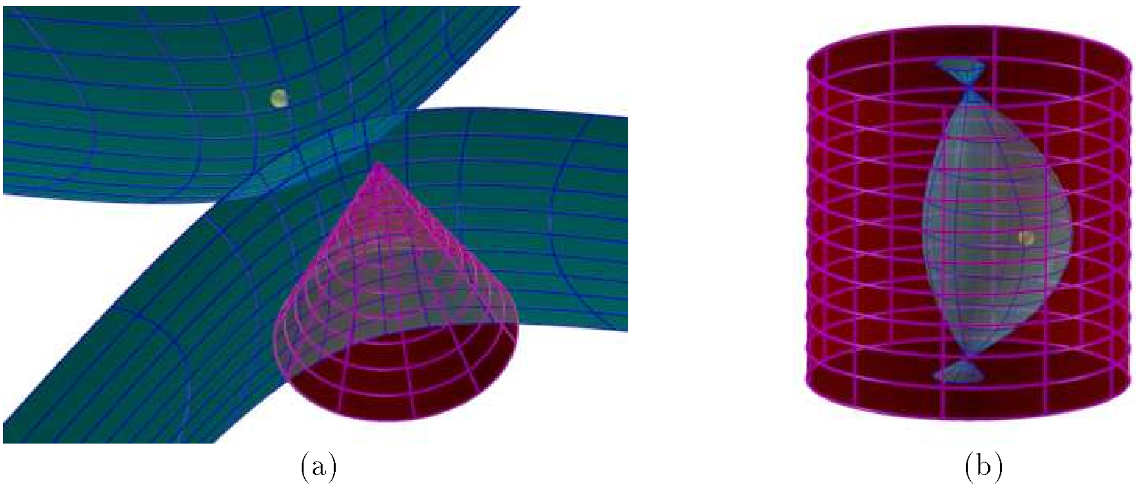
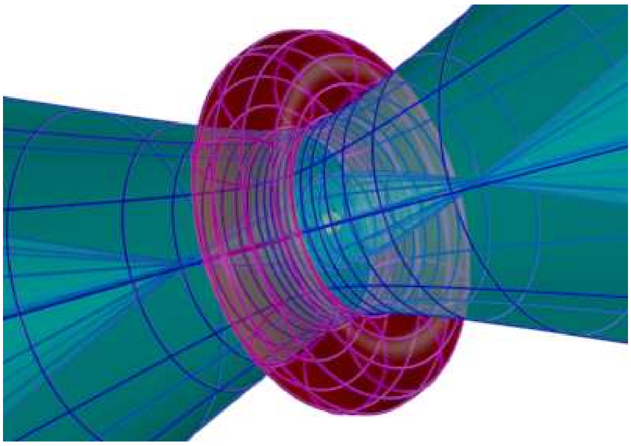


Figure 3: Views of the bisector surface between a point and a cone and that between a point and a cylinder. In (a), the point is outside the cone forming two infinite sheets due to poles in the bisector's parameterization, while in (b), the point is inside the cylinder.

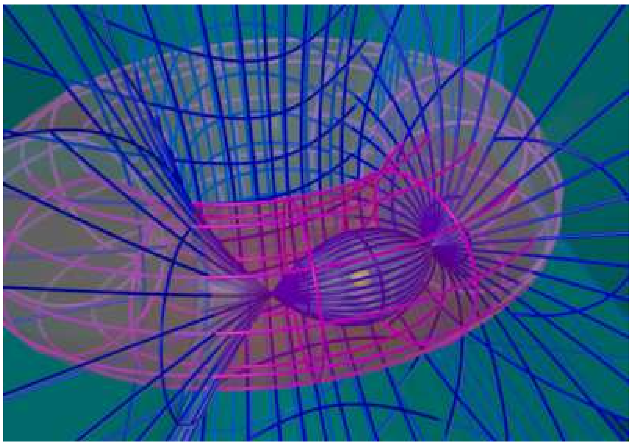
and surfaces with rational offsets, which includes the Pythagorean Hodograph (PH) curves as an important subclass of all polynomial curves with rational offsets [6]. Pottmann et al. [10] showed that all non-developable rational ruled surfaces also belong to this class of rational surfaces with rational offsets. Consequently, our result can be applied to a wide variety of rational curves/surfaces for the construction of their rational bisector curves/surfaces with circles/spheres.

In Figure 5, notice that the bisector surface consists of three components, where two adjacent components meet at a self-intersection point of the bisector surface. Only the middle part belongs to the true bisector surface since the other two components are closer to the canal surface than the sphere. The elimination of these redundant components is called *trimming*.

Given a point and an arbitrary surface, their trimmed bisector surface bounds a convex region that contains the point [4]. Similarly, the trimmed bisector surface between a sphere and a surface also bounds a convex region that contains the sphere since it is essentially the same as the bisector surface between the sphere center and an offset surface. Given a surface and a multi-layer of concentric spheres, there is a corresponding multi-



(a)



(b)

Figure 4: Two views of the bisector surface between a point and a torus. In (a), the point is at the center of the torus, while in (b), the point is inside the torus.

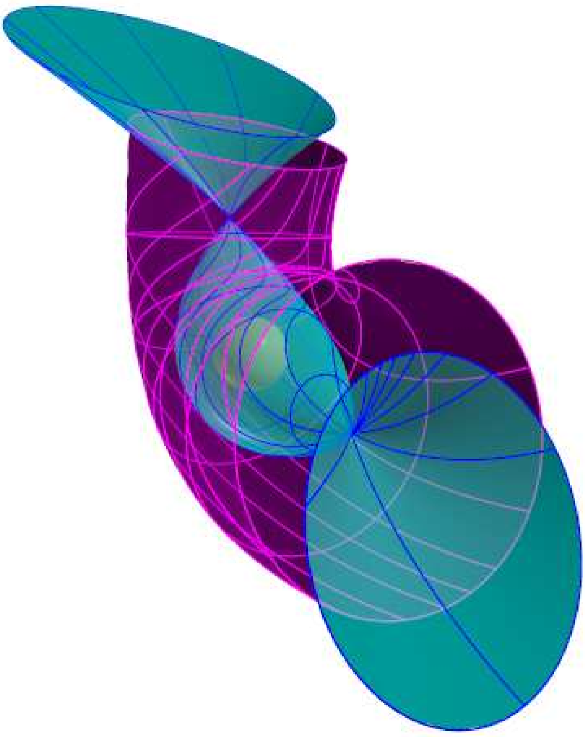


Figure 5: A view of the bisector surface between a sphere and a canal surface. The bisector was computed by offsetting the canal surface and then computing the bisector between the center of the sphere and the offset of the canal surface.

layer of convex trimmed bisector surfaces. When the given surface is a surface with rational offsets, we can generate a level set of convex trimmed rational bisector surfaces that propagate from a point. Moreover, each bisector surface has its rational parameterization inherited from that of the given rational surface. This capability enables us to generate a new rational local coordinate system in the neighborhood of the given point.

In general, the trimmed bisector surface may consist of many subpatches bounded by the self-intersection curve of the untrimmed bisector. Consequently, an exact trimming procedure is non-trivial to develop since it is difficult to determine *a priori* how many subpatches the trimmed bisector has. Based on the convexity of the region bounded by a trimmed bisector, we can develop a subdivision-based method that samples points on the bisector surface, construct supporting planes at the points, construct a convex polyhedron bounded by these planes, and trim out the bisector surface components that belong to the exterior of the convex polyhedron. This procedure can be repeated to the surface regions where trimmings have occurred. We defer more details of this approach to a future work.

Conclusion

In this article, we have shown that the bisector between a point and a rational surface is a rational surface. This result implies that the bisector between a sphere and a rational surface with rational offsets is also a rational surface. The class of rational surfaces with rational offsets includes various different types of surfaces: planes, spheres, cylinders, cones, torii, Dupin cyclides, rational canal surfaces, non-developable rational ruled surfaces, etc. Recent development of Pythagorean Hodograph

(PH) space curves also accelerates the advance of surface design techniques for rational sweep surfaces with rational offsets. Consequently, the bisector construction scheme we have introduced in this article has much potential for applications in practice.

Acknowledgment

We are thankful for Martin Peternell and Helmut Pottmann for providing us with their Maple program for constructing rational canal surfaces (and rational offsets also). This program was used for the example shown in Figure 5.

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