

Linearizing the Area and Volume Constraints

Gershon Elber
Computer Science Department
Technion
Haifa 32000, Israel
Email: gershon@cs.technion.ac.il

Abstract

Positional, tangential, and orthogonality constraints are all linear and can be easily incorporated into freeform curve and surface editing and manipulation environment. In this paper, we show that the area and volume constraints that are traditionally considered non linear and hence difficult to handle, can indeed be reformulated as linear constraints and similarly incorporated with ease into editing environment.

Keywords: CAD, Curves & Surfaces, Geometric Modeling, Transfinite constraints.

1 Introduction

The synergy of a precise yet intuitive and interactive editing capabilities of freeform curves and surfaces in geometric cad systems has been an elusive task for the geometric design community, for a long period of time. While freeform curves were introduced to the computer graphics world about three decades ago, the editing process of freeform geometry continues to be considered a difficult task. Contemporary modeling environments are capable of handling a limited set of constraints, mostly linear, in the context of interactive design. Hence, it is crucial to identify the set of linear constraints that one can employ, in freeform design.

In [Welc92], a surface editing system that satisfies zero dimensional constraints such as positions, tangents and normals, has been presented. The constraints, being linear, are efficiently solved, allowing for the interactive manipulation of the freeform geometry. [Welc92] also considers *transfinite constraints* where the constraints might have a non zero dimensionality. While some cases might be of finite dimension, such as the containment of a polynomial curve in a polynomial surface when posed as a composition, other cases might necessitate an approximation.

The satisfaction of nonlinear constraints is significantly more difficult than the satisfaction of linear constraints due to the imposed computational demands. Nevertheless and mostly due to their importance, non linear constraints are considered in a whole variety of cases. Few examples of nonlinear constraints in use include second order differential constraints such as convexity [Sapi95], enclosed volume [Rapp96], and first and second order fairing constraint, typically in the form of strain and stress surface shape optimization functionals [Welc92].

While highly intensive computationally, a successful use of first and second differential order constraints, in real time, is reported in [Plav98], for bicubic Bézier surfaces. An interactive system that supports real time surface manipulation with convexity/developability constraints is discussed, with the aid of a careful pre-computation of the curvature fields.

In this paper, we show that the enclosed area of a closed parametric curve and the enclosed volume of a closed parametric surface could also be posed as linear constraints. Our implementation as well as the examples shown as part of this work are based on the IRIT [Irit97] solid modeling system that is developed at the Technion, Israel Institute of Technology.

This paper is organized as follows. Section 2 discusses the area constraint for closed parametric curves. Some examples that demonstrate the use of the area constraint in an interactive system are presented in Section 3. We consider the extensions of this work to freeform surfaces in Section 4, and, finally, we conclude in Section 5.

2 Area Constraint of a Closed Planar Parametric Curve

The enclosed area or volume of a closed parametric curve or a closed parametric surface, respectively, were considered, for example, in the context of vision [Eber91] as well as geometric modeling [Ocho98, Rapp96]. Presented as a non linear problem, approximation and/or optimization methods are typically employed toward the computation of the enclosed property.

Examine the area equation of a closed planar freeform parametric curve, $C(t) = (x(t), y(t))$. Employing Green's theorem, the (signed) area, \mathcal{A} , enclosed by $C(t)$ equals (See, for example [Eber91, Ocho98]),

$$\mathcal{A} = \frac{1}{2} \oint -x'(t)y(t) + x(t)y'(t)dt. \quad (1)$$

Being planar, the above also equals $\frac{1}{2} \oint |C(t) \times C'(t)|dt$, with $|\cdot|$ denoting a determinant. The

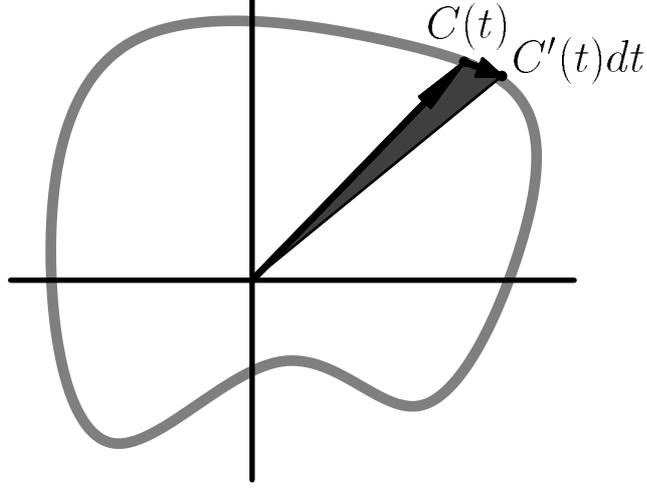


Figure 1: The area of a closed parametric curve. The differential area in gray equals to $\frac{1}{2}|C(t) \times C'(t)|dt$ and the enclosed area by the curve is the result of integrating this differential area over the entire parametric domain of the curve. See Equation (1).

quadratic expression of Equation (1) resulted in attempts to employ approximation and/or non linear optimization techniques to satisfy the area (and volume) constraints.

A geometric interpretation of Equation (1) can be found in Figure 1. Herein, we are interested in evaluating this equation as efficiently as possible when $C(t)$ is a B-spline curve. Moreover, we are pursuing this computation in the context of (linear) constraint satisfaction, in real time interaction.

Assume $C(t)$ is indeed a B-spline curve, $C(t) = \sum_{i=0}^{n-1} P_i B_{i,k}(t)$, where $P_i = (x_i, y_i)$ and $B_{i,k}(t)$ is the i 'th B-spline basis function of order k . Then, Equation (1) reduces to,

$$\begin{aligned} 2\mathcal{A} &= \oint -\sum_i x_i B'_{i,k}(t) \sum_j y_j B_{j,k}(t) + \sum_i x_i B_{i,k}(t) \sum_j y_j B'_{j,k}(t) dt \\ &= \sum_i x_i \sum_j y_j \oint -B'_{i,k}(t) B_{j,k}(t) + B_{i,k}(t) B'_{j,k}(t) dt. \end{aligned} \quad (2)$$

Now, area Equation (2) can be rewritten as the *bilinear form* of,

$$\begin{aligned} 2\mathcal{A} &= \begin{bmatrix} x_0 & x_1 & \cdots & x_{n-1} \end{bmatrix} \begin{bmatrix} \phi_{0,0} & \phi_{0,1} & \cdots & \phi_{0,n-1} \\ \phi_{1,0} & \phi_{1,1} & \cdots & \phi_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n-1,0} & \phi_{n-1,1} & \cdots & \phi_{n-1,n-1} \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} \\ &= \mathbf{X} \mathbf{\Phi} \mathbf{Y}, \end{aligned} \quad (3)$$

where

$$\phi_{ij} = \oint -B'_{i,k}(t) B_{j,k}(t) + B_{i,k}(t) B'_{j,k}(t) dt$$

$$\begin{aligned}
 &= \oint - (k-1) \left(\frac{B_{i,k-1}(t)}{t_{i+k-1} - t_i} - \frac{B_{i+1,k-1}(t)}{t_{i+k} - t_{i+1}} \right) B_{j,k}(t) \\
 &\quad + (k-1) \left(\frac{B_{j,k-1}(t)}{t_{j+k-1} - t_j} - \frac{B_{j+1,k-1}(t)}{t_{j+k} - t_{j+1}} \right) B_{i,k}(t) dt.
 \end{aligned} \tag{4}$$

Equation (3) sheds some light on our objectives. Assume the y_i coefficients of the constrained curve are fixed. Then, the area constraint is linear in the x_i coefficients! Similarly, if the x_i coefficients of the constrained curve are fixed, the area constraint is linear in the y_i coefficients.

This crucial view of the area equation allows us not only to enforce a prescribed area as a linear constraint, but also to pre-compute all the coefficients of the Φ matrix with the aid of products [Elbe92, Mork91] and integrals of B-spline [Fari97, Kazi97] functions. A typical editing session of a freeform shape starts with a prescription of the specific function space, Ψ , or knot sequence. Only then, the shape is modified, for example via a direct or a control point select-and-drag operation. As the user is dragging a point on the curve, numerous drag (mouse) events are generated. The role of the x_i and the y_i coefficients is alternated in every such drag event, fixing the set of coefficients along one axis and using the other axis' set to satisfy the area constraint. As a result, the global affect of the x and y axes on the area evens out.

2.1 Area Constraint of Linear B-spline Curves

It is interesting to examine this derived area constraint in the context of linear B-spline curves. Then, the B-spline curve is reduced to a polygon and hence we expect the constraint to reduce to the equation of the area of a polygon with n vertices. For the linear B-spline case where $k = 2$, Equation (4) becomes,

$$\phi_{ij} = \oint - \left(\frac{B_{i,1}(t)}{t_{i+1} - t_i} - \frac{B_{i+1,1}(t)}{t_{i+2} - t_{i+1}} \right) B_{j,2}(t) + \left(\frac{B_{j,1}(t)}{t_{j+1} - t_j} - \frac{B_{j+1,1}(t)}{t_{j+2} - t_{j+1}} \right) B_{i,2}(t) dt. \tag{5}$$

Clearly if $|i - j| \geq 2$ and due to the final support of the B-spline basis functions, Equation (5) is zero. Moreover, if $i = j$ and due to the antisymmetry of the integrand in Equation (5), ϕ_{ii} is also zero. Hence, we only need to derive $\phi_{i,i\pm 1}$:

$$\begin{aligned}
 \phi_{i,i+1} &= \oint - \left(\frac{B_{i,1}(t)}{t_{i+1} - t_i} - \frac{B_{i+1,1}(t)}{t_{i+2} - t_{i+1}} \right) B_{i+1,2}(t) + \left(\frac{B_{i+1,1}(t)}{t_{i+2} - t_{i+1}} - \frac{B_{i+2,1}(t)}{t_{i+3} - t_{i+2}} \right) B_{i,2}(t) dt \\
 &= \oint \frac{B_{i+1,1}(t)}{t_{i+2} - t_{i+1}} B_{i+1,2}(t) + \frac{B_{i+1,1}(t)}{t_{i+2} - t_{i+1}} B_{i,2}(t) dt
 \end{aligned}$$

$$\begin{aligned}
 &= \oint \frac{B_{i+1,1}(t)}{t_{i+2} - t_{i+1}} (B_{i+1,2}(t) + B_{i,2}(t)) dt \\
 &= \int_{t_{i+1}}^{t_{i+2}} \frac{1}{t_{i+2} - t_{i+1}} dt \\
 &= 1.
 \end{aligned} \tag{6}$$

$\phi_{ij} = -\phi_{ji}$ (See Equation (4)) or Φ is an antisymmetric matrix. Hence, $\phi_{i,i-1} = -1$, and we have shown that the area of a closed polygon $\mathcal{P} = \{P_i\}_{i=0}^{n-1}$, $P_i = (x_i, y_i)$ equals,

$$\mathcal{A} = \frac{1}{2} \begin{bmatrix} x_0, x_1, \dots, x_{n-1}, x_0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ -1 & 0 & 1 & 0 & \dots & 0 \\ 0 & -1 & 0 & 1 & \dots & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & -1 & 0 & 1 \\ 0 & 0 & 0 & \dots & -1 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \\ y_0 \end{bmatrix}, \tag{7}$$

which is the same as,

$$\mathcal{A} = \frac{1}{2} \left(\begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \dots + \begin{vmatrix} x_{n-1} & x_0 \\ y_{n-1} & y_0 \end{vmatrix} \right), \tag{8}$$

as, for example, in [Pear90]. This linear Bspline case allows one to edit closed planar polygonal domains while coercing the enclosed area of the polygon to be the same throughout the editing process, via a *bilinear constraint* over the vertices of the polygon.

3 Examples

The area constraint, being linearized, was incorporated into a freeform editing system that supports linear constraints. All the examples presented in this section are the results of interactive sessions where the user directly manipulate nonuniform B-spline curves. During the interaction, a fixed area constraint and possibly additional linear constraints are imposed. In all examples, the space of the curve was not modified during the editing process and the stationary Φ matrix (see Equation (3)) was pre-computed once, in a fraction of a second on a modern workstation.

The cross section in Figure 2, possibly of a fuselage of a plane, is constrained to present a constant area, and hence eventually a fixed volume of the plane. Furthermore, and for obvious reasons, the curve is also constrained to be Y -symmetric. Shown in the figure are several cross sections that were all derived in few seconds of user interaction time from the original curve, in gray, while preserving a fixed area constraint.

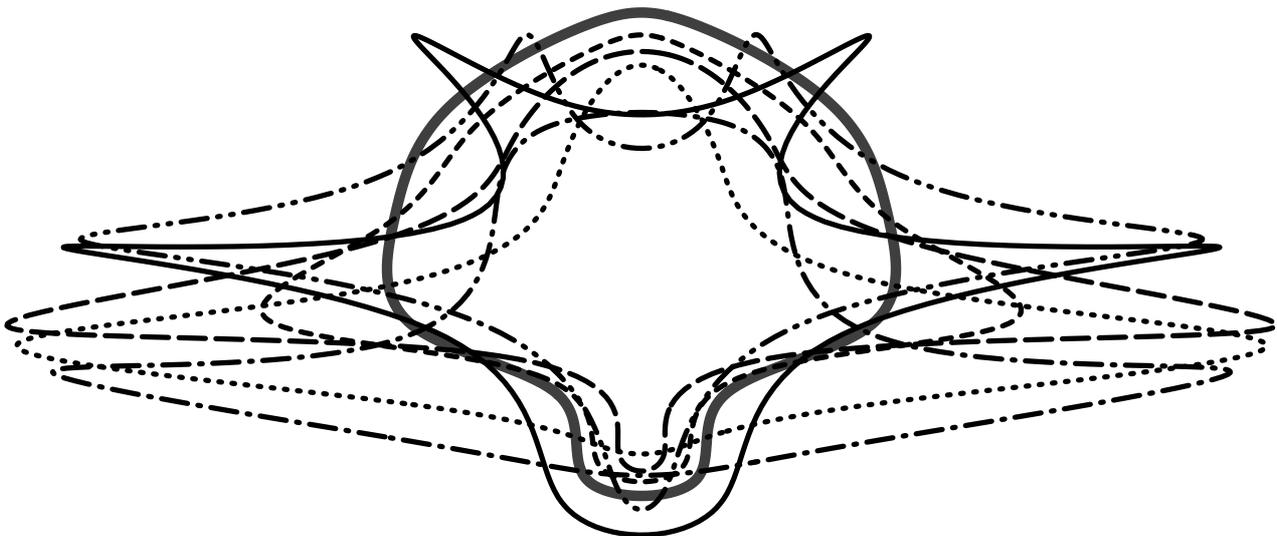


Figure 2: In gray, a periodic planar B-spline cross section is shown that is constrained to be Y -symmetric as well as to present a fixed area. All the other cross sections were derived from it in few seconds via direct manipulation while the area as well as the Y -symmetry are preserved. The curve is a cubic periodic curve with twelve control points.

The closed cross sections in Figure 3 are also constrained to present a constant area. A single point is selected and dragged, resulting in the motion of the entire shape due to the preservation of this area constraint. Furthermore, other linear constraints could be simultaneously applied as is demonstrated in Figure 3 (c).

4 Extensions to Freeform Surfaces

The presented view of an enclosed area as a bilinear constraint could also be extended to handle the enclosed volume of freeform surfaces. The decomposition of the area constraint into a bilinear form is similarly extendible to the volume enclosed by a parametric B-spline surface. Following [Ocho98], the signed volume V , enclosed by parametric surface $S(u, v)$ equals,

$$V = \int_U z \left(\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u} \right) dudv, \quad (9)$$

where U is the parametric domain of

$$S(u, v) = \sum_i \sum_j P_{ij} B_{i,k_u}(u) B_{j,k_v}(v),$$

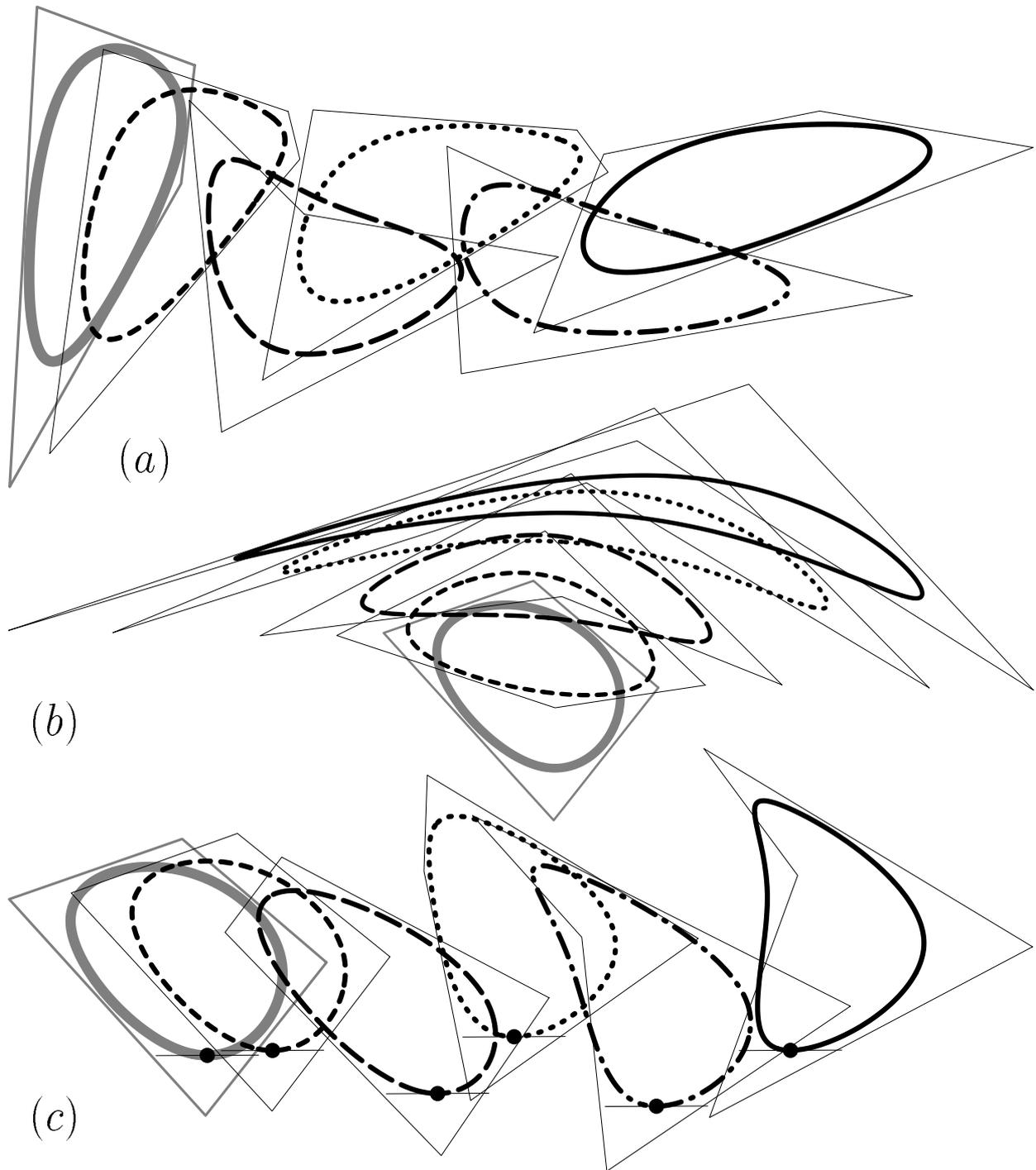


Figure 3: In (a), a periodic planar cubic B-spline curve in gray is dragged to the right in several steps while its area is preserved. The other curves are the result of dragging a single point on the curve, while the rest of the curve is following along, to preserve the fixed area constraint. In (b), the bottom control points of a periodic planar cubic B-spline curve in gray is pushed upward in several steps while its area is again, preserved. In (c), a tangent constraint is added at the bottom of the cubic shape. This tangent constraint (in addition to the area constraint) introduces the effect of slope preservation while dragged to the right.

and $P_{ij} = (x_{ij}, y_{ij}, z_{ij})$. Then,

$$\begin{aligned}
 V &= \int_U \sum_{l_u} \sum_{l_v} z_{l_u, l_v} B_{l_u, k_u}(u) B_{l_v, k_v}(v) \\
 &\quad \left(\sum_{i_u} \sum_{i_v} x_{i_u, i_v} B'_{i_u, k_u}(u) B_{i_v, k_v}(v) \sum_{j_u} \sum_{j_v} y_{j_u, j_v} B_{j_u, k_u}(u) B'_{j_v, k_v}(v) \right. \\
 &\quad \left. - \sum_{i_u} \sum_{i_v} x_{i_u, i_v} B_{i_u, k_u}(u) B'_{i_v, k_v}(v) \sum_{j_u} \sum_{j_v} y_{j_u, j_v} B'_{j_u, k_u}(u) B_{j_v, k_v}(v) \right) dudv \\
 &= \sum_{i_u} \sum_{i_v} x_{i_u, i_v} \sum_{j_u} \sum_{j_v} y_{j_u, j_v} \sum_{l_u} \sum_{l_v} z_{l_u, l_v} \\
 &\quad \int_U B_{l_u, k_u}(u) B_{l_v, k_v}(v) \\
 &\quad \left(B'_{i_u, k_u}(u) B_{i_v, k_v}(v) B_{j_u, k_u}(u) B'_{j_v, k_v}(v) \right. \\
 &\quad \left. - B_{i_u, k_u}(u) B'_{i_v, k_v}(v) B'_{j_u, k_u}(u) B_{j_v, k_v}(v) \right) dudv. \tag{10}
 \end{aligned}$$

Hence, the volume enclosed by a parametric B-spline surface, in Equation (9), is reducible to a tri-linear form in the x_{i_u, i_v} , y_{j_u, j_v} and z_{l_u, l_v} coefficients of the surface S . In a similar way to Equation (3), one can a-priori compute the integral of the products of the basis functions in Φ , with

$$\begin{aligned}
 \phi_{i_u, i_v, j_u, j_v, l_u, l_v} &= \int_U B_{l_u, k_u}(u) B_{l_v, k_v}(v) \\
 &\quad \left(B'_{i_u, k_u}(u) B_{i_v, k_v}(v) B_{j_u, k_u}(u) B'_{j_v, k_v}(v) \right. \\
 &\quad \left. - B_{i_u, k_u}(u) B'_{i_v, k_v}(v) B'_{j_u, k_u}(u) B_{j_v, k_v}(v) \right) dudv.
 \end{aligned}$$

With the aid of this tri-linear form in the coefficient of the surface, during the surface interactive manipulation, one is required to solve for the linear constraint of the volume in either the x , the y , or the z coefficients of the surface, in alternating order.

5 Conclusions

We have shown and demonstrated that the area constraint of a freeform parametric curve could be reformulated as a bilinear form and then incorporated as a linear constraint into contemporary freeform editing environments. Further, we have shown a similar reformulation for freeform parametric surfaces, constraining the enclosed volume. The presented approach could be extended with ease to support a closed shape composed of several parametric curves or even a closed volume composed of several parametric surface, summing the area or volume constraints, over them all.

We are hopeful that the approach presented in this work that decomposes a multiplicative higher order constraint into a product of independent linear forms could be applied to other constraints as well, including other dimensional zero moments.

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